Notes on Heat Transfer in a Layered Longeron

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15 May 2006

1 Preliminary Comments

We are interested in a dynamic model for heat transfer in a longeron of a space truss structure. The longeron is a right circular cylinder and is composed of several layers (see Figure 1). We treat certain layers as thin cylindrical shells and account for the heat transfer between adjacent layers and also to the surroundings.

2 Basic Ideas

For the $i^{th}$ thin layer (thickness $\delta_i$) we have an unsteady conduction model

$$(\rho C_p) \frac{\partial T_i}{\partial t} = \nabla_n^2 T_i + \frac{S_i}{\delta_i}, \quad \text{on } \Omega = [-L/2, L/2] \times [-\pi, \pi], \ t > 0,$$  \hspace{1cm} (1)

where the ‘non-isotropic differential operator’ accounts for the fact that the thermal conductivity differs in the axial ($z$) direction and in the circumferential ($\theta$). In particular we have

$$\nabla_n T_i = \left[ \sqrt{k_a} \hat{e}_z \frac{\partial}{\partial z} + \sqrt{k_c} \hat{e}_\theta \frac{\partial}{\partial \theta} \right] T_i$$ \hspace{1cm} (2)

so that

$$\nabla_n^2 T_i = k_a \frac{\partial^2 T_i}{\partial z^2} + \frac{k_c}{R^2} \frac{\partial^2 T_i}{\partial \theta^2}.$$  

$S_i$ is a (surface flux, W/m$^2$) source term and accounts for energy transfer to the surface ($S_i > 0$). Much of the modeling effort is concerned with describing the nature of these source terms.
3 Formal Model

The layers in our model are:

- Inner Bladder
- Rigidizable/Inflatable (RI) material
- Kapton Heater
- Multi-Layer Insulation (MLI)

We include the unsteady conduction model (2) for the first two layers (only). In particular, our current model does not include thermal capacitance for either the Kapton Heater or for the MLI.

3.1 Inner Surface of Bladder Layer

To reflect construction features in the application (for example, the interior of the inner-layer is in contact with three distinct thermal masses) we decompose the region $\Omega$ into a covering set $\{\Omega_1, \Omega_2, \Omega_3\}$, with $\cup_{k=1}^{n}\Omega_k = \Omega$, and $\Omega_k \cap \Omega_\ell = \emptyset$, $k \neq \ell$. With each region we associate a thermal mass that
can exchange energy with the adjacent region \( (\Omega_k) \) of the inner-layer. In particular, we have

\[
\begin{align*}
\Omega_1 &= [-L/2, -L/2 + \Delta] \times [0, 2\pi], \\
\Omega_2 &= [-L/2 + \Delta, L/2 - \Delta] \times [0, 2\pi], \\
\Omega_3 &= [L/2 - \Delta, L/2] \times [0, 2\pi].
\end{align*}
\]

Regions \( \Omega_1 \) and \( \Omega_3 \) account for the bond region wherein the cylinder material is joined to the end-cap. The region bounded by \( \Omega_2 \) contains the inflation gas \( (N_2) \). Thus, part of the source term for the inner-most layer of the longeron is given by

\[
S_1(t, z, \theta) = C_g \left( T_g(t) - T_1(t, z, \theta) \right) \chi_g(z, \theta),
\]

where \( \chi_g \) is the characteristic function of the region \( \Omega_g = \Omega_2 \). Here \( C_g \) is a (constant) conductance parameter for the gas region, and \( T_g \) is the bulk temperature of this mass. This temperature evolves according to

\[
C_g \frac{dT_g(t)}{dt} = Q_g(t),
\]

where \( Q_g \) is the net thermal energy-rate into the gas.

**Internal Radiation**

The \( \Omega_2 \) region of the inner bladder can radiantly exchange energy with itself and with the metal end-caps. On a radiating surface there are two fields of interest [5, see Section 3-3]

- \( H \) – the irradiation is the arriving radiant flux; and,
- \( B \) – the radiosity is the departing radiant flux.

The net heat flux to a surface is the difference between the energy absorbed and the energy emitted

\[
S = \alpha H - \varepsilon \sigma T^4,
\]

where \( \sigma T^4 \) is the ideal black-body radiation (Stefan-Boltzman Law), \( \varepsilon \) is the emissivity of the surface, and \( \alpha \) is the absorptivity. It follows that;

\[
B = \varepsilon \sigma T^4 + \rho H,
\]

(\( \rho \) is the reflectivity). Assuming that \( (\alpha + \rho = 1) \) one can eliminate \( H \) leading to

\[
S = \frac{\alpha}{1 - \alpha} \left[ B - \frac{\varepsilon}{\alpha} \sigma T^4 \right].
\]
We write a flux-balance equation (4) for the interior of the cylinder bladder. In this case, an element on the interior surface receives radiation from other (visible) points on the interior surface and from the end-cap regions. We have assumed that each (metallic) end-cap (left \( l \), and right \( r \)) is characterized by a single temperature. Thus, the radiosity distribution for the cylinder at time \( t \) and location \( \vec{r}_c \) must satisfy the integral equation:

\[
B_c(t, \vec{r}_c) = \epsilon_c \sigma T_c^4(t, \vec{r}_c) + (1 - \alpha_c) \int_{\Gamma_c} K^{(c,c)}(\vec{r}_c, \vec{p}_c) B_c(t, \vec{p}_c) \, d\Gamma_c \\
\theta + (1 - \alpha_c) \left[ \int_{\Gamma_l} K^{(l,c)}(\vec{r}_c, \vec{p}_l) \epsilon_l \sigma T_l^4(t) \, d\Gamma_l + \int_{\Gamma_r} K^{(r,c)}(\vec{r}_c, \vec{p}_r) \epsilon_r \sigma T_r^4(t) \, d\Gamma_r \right].
\]

In this expression the kernel functions \( K^{(p,q)} \) characterize certain view factors (geometric factors, shape factors) [5, 4].

\[
F_{i\rightarrow j} \Delta = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} \, dA_i dA_j.
\]

Each of the direction cosines can be expressed as an inner-product of a surface normal (\( \hat{\eta} \)) and the vector between points on the surfaces (see Figure 2).

Hence, we write the kernel function as

\[
K(\vec{r}_i, \vec{r}_j) = \frac{[\hat{\eta}_i \cdot (\vec{r}_j - \vec{r}_i)][\hat{\eta}_j \cdot (\vec{r}_i - \vec{r}_j)]}{\pi \| (\vec{r}_j - \vec{r}_i) \|^4}.
\]

We now apply these ideas to the kernel function \( K^{(c,c)} \). Points on the interior of the cylinder (radius \( a \)) can be characterized by coordinates \((z, \theta)\),

\[
\vec{r}_c(z, \theta) = (a \cos \theta, a \sin \theta, z),
\]
while the inward unit normal is given by
\[ \hat{n}_c = (-\cos \theta, -\sin \theta, 0) . \]

With these expressions for points on the interior cylindrical surface, the kernel function (7) for cylinder-to-cylinder radiant exchange is
\[ K^{(c,c)}(z_1, \theta_1, z_2, \theta_2) = \frac{a^2 [1 - \cos(\theta_1 - \theta_2)]^2}{\pi [2a^2 (1 - \cos(\theta_1 - \theta_2)) + (z_1 - z_2)^2]^2} \] (8)

For the End-Cap/Bladder interaction, points on the interior of the (left) end-cap can be characterized by coordinates \((r, \theta_l)\),
\[ \vec{r}_c(r, \theta_l) = (r \cos \theta_l, r \sin \theta_l, -L') , \]
while the inward unit normal is given by
\[ \hat{n}_l = (0, 0, 1) . \]

Here we have used \(L' = L/2 - \Delta\). With these expressions for points on the interior surfaces, the kernel function (7) for cylinder-to-endcap radiant exchange is
\[ K^{(l,c)}(z, \theta_c, r, \theta_l) = \frac{(z + L')(a - r \cos(\theta_c - \theta_l))}{\pi [(L' - z)^2 + r^2 + a^2 - 2 a r \cos(\theta_c - \theta_l)]^2} \] (9)

\[ K^{(r,c)}(z, \theta_c, r, \theta_r) = \frac{(z + L')(a - r \cos(\theta_c - \theta_r))}{\pi [(L' - z)^2 + r^2 + a^2 - 2 a r \cos(\theta_c - \theta_r)]^2} \] (10)

Using this coordinate representation, the integral equation (6) is written
\[ B_c(t, z, \theta) = \epsilon_c \sigma T^4_c(t, z, \theta) + (1 - \alpha_c) \int_{-\pi}^{\pi} \int_{-L'}^{L'} K^{(c,c)}(z, \theta, z_2, \theta_2) B_c(t, z_2, \theta_2) \, dz_2 \, d\theta_2 \]
\[ + (1 - \alpha_c) \left[ \epsilon_l \sigma T^4_l(t) \int_{-\pi}^{\pi} \int_{0}^{a} K^{(l,c)}(z, \theta, r, \theta_l) \, dr \, d\theta_l \right] \]
\[ \epsilon_r \sigma T^4_r(t) \int_{-\pi}^{\pi} \int_{0}^{a} K^{(r,c)}(z, \theta, r, \theta_r) \, dr \, d\theta_r \]

Note that the latter two integrals are independent of the radiosity. We define the function
\[ g^{(l,c)}(z, \theta) \triangleq \int_{-\pi}^{\pi} \int_{0}^{a} K^{(l,c)}(z, \theta, r, \theta_l) \, dr \, d\theta_l , \] (11)
and similarly for \( g^{(r,c)}(z, \theta) \). The integral equation for the radiosity \( B_c \) is then written as

\[
B_c(t, z, \theta) = (1 - \alpha_c) \int_{\pi}^{\pi} \int_{-L'}^{L'} K^{(c,c)}(z, \theta, z_2, \theta_2) B_c(t, z_2, \theta_2) \, dz_2 \, d\theta_2 \\
+ \epsilon_c \sigma T_c^4(t, z, \theta) + (1 - \alpha_c) \left[ \epsilon_l \sigma T_1^4(t) g^{(l,c)}(z, \theta) + \epsilon_r \sigma T_r^4(t) g^{(r,c)}(z, \theta) \right].
\]

(12)

With the cylinder radiosity function \( B_c \) from (12), the radiant contribution to the surface flux at the inner bladder is given by (5).

### 3.2 Interface Between Bladder and RI

Energy flux from the bladder layer to the RI layer is modeled as

\[
S_b(t, z, \theta) = C_b(T_1(t, z, \theta) - T_2(t, z, \theta)).
\]

If we interpret \( T_i \) as the temperature at the mid-plane of the \( i \)-th layer, then the (constant) conductance \( C \) models the conduction through one-half of layer 1, a contact resistance between the layers, and conduction through one-half of layer 2.

\[
C_b = \left[ \frac{(\delta_1/2)}{k_1} + R_c + \frac{(\delta_2/2)}{k_2} \right]^{-1}.
\]

\( R_c \) is a ‘contact resistance’ for a unit area. Alternate expressions, reflecting the cylindrical geometry, are possible for the first and last terms. However, given the expected uncertainty in the parameter values, the proposed model is likely adequate.

### 3.3 Kapton Heater Layer

We model the Kapton Heater as a uniform source of power

\[
S_H(t, z, \theta) = \frac{P(t)}{A_c},
\]

where \( A_c \) is the surface area of the heater \( A_c = 2 \pi a L \).
3.4 MLI Layer

The MLI layer admits thermal energy transmission from its inner surface to the outer surface through an inter-layer radiative transfer process. The mathematical model for the associated heat flux is

\[ q = \sigma \epsilon^* [T_i^4 - T_o^4] \]

where \( T_i \) is the temperature of the inner layer, \( T_o \) is the temperature of the outer layer, and \( \epsilon^* \) is the effective emissivity of the MLI [2, see pp 147-157]. In the space application, heat transfer from the external surface is via radiation. Specifically, a portion of arriving (solar + Earth) radiation is absorbed

\[ S_a = \alpha_o H_a \]

where \( \alpha_o \) is a radiation absorption coefficient, and \( H_a \) is the arriving radiant flux (irradiation). Additionally, energy is radiated away according to the usual Stefan-Boltzman law

\[ S_B = \sigma \epsilon_o T_o^4 \]

An energy balance on the outer MLI surface yields

\[ q + S_a - S_B = 0 \]

Using Equation (13) to solve for \( T_o \), the energy balance equation leads to

\[ q = \frac{\sigma \epsilon_o T_i^4 - \alpha_o H_a}{1 + \frac{\epsilon_o}{\epsilon^*}} \].

3.5 Summary of Source Terms

The temperature distributions in the bladder \( (T_1) \) and the RI \( (T_2) \) evolve according to the differential equation (1) with source terms. The building blocks for these source terms have been discussed above. We now assemble these for layers 1 and 2 on the various subdomains.

3.5.1 Bladder: Layer 1

- \( \Omega_1 \):
  On this region the bladder is in contact with the left inner ring (bottom) and the RI layer. Thus, on this region the energy flux to the bladder is given by:

\[ S_1(t, z, \theta) = C_\ell (\overline{T}(t) - T_1(t, z, \theta)) + C_b (T_2(t, z, \theta) - T_1(t, z, \theta)) \].

(15)
• \(\Omega_2\):
  On \(\Omega_2\) the bladder is in contact with the inflation gas and itself (radiation) and the RI layer (top). Thus, on \(\Omega_2\)
  \[
  S_1(t, z, \theta) = \frac{\alpha_b}{1 - \alpha_b} \left[ B_c(t, z, \theta) - \frac{\sigma \epsilon_b}{\alpha_b} T_1^4(t, z, \theta) \right] + \\
  C_g \left( T_g(t) - T_1(t, z, \theta) \right) + C_b \left( T_2(t, z, \theta) - T_1(t, z, \theta) \right) .
  \] (16)
  Note that the surface \(c\) in (12) is the bladder; the temperature field is given by \(T_1(t, \cdot, \cdot, \cdot)\). Additionally, the left and right bulk temperatures are associated with the inner end-cap and are denoted by \(T_{i\ell}\) and \(T_{ir}\), respectively.

• \(\Omega_3\):
  Here the bladder is in contact with the right inner ring (bottom) and the RI layer. Thus,
  \[
  S_1(t, z, \theta) = C_r \left( T_{ir}(t) - T_1(t, z, \theta) \right) + C_b \left( T_2(t, z, \theta) - T_1(t, z, \theta) \right) .
  \] (17)

3.5.2 Rigidizable/Inflatable: Layer 2

• \(\Omega_1\):
  On this region the RI material is conductively coupled to the outer left ring (above), and to the bladder (below) leading to
  \[
  S_2(t, z, \theta) = C_{H1} \left( T_{ol}(t) - T_2(t, z, \theta) \right) + C_b \left( T_1(t, z, \theta) - T_2(t, z, \theta) \right) .
  \] (18)
  Note that the first term is a heat drain from the outer ring.

• \(\Omega_2\):
  Here the RI layer is conductively coupled to the bladder (bottom) and radiatively coupled to the heater and MLI (above). A pictorial description of our model for radiative exchange with the heater/MLI is shown in Figure 3. In particular, if the potential at each node is computed as \(\sigma T^4\), then for the conductance parameters in Figure 3 we have
  \[
  C_M = \epsilon^* , \\
  C_u = \left( \frac{1}{\epsilon_H} + \frac{1}{\epsilon_i} - 1 \right)^{-1} , \\
  C_l = \left( \frac{1}{\epsilon_H} + \frac{1}{\epsilon_{RI}} - 1 \right)^{-1} .
  \]
The heat flows $q_u$ and $q_l$ are computed as:

$$q_u = C_u (\sigma T_H^4 - \sigma T_i^4)$$
$$q_l = C_M (\sigma T_i^4 - \sigma T_o^4)$$

There are three unknown potentials, viz. $\sigma T_H^4$, $\sigma T_i^4$, $\sigma T_o^4$, whereas instantaneous values of $\sigma T_R^4$, $q_H$, and $H_a$ are known. Equating the two expressions for $q_u$ provides one linear algebraic equation in the unknown potentials. Neglecting thermal capacitance in the heater one has $q_H = q_u + q_l$, and this adds a second linear equation. Finally, an energy balance at the $T_o$ node provides a third equation, namely:

$$\epsilon^* (\sigma T_i^4 - \sigma T_o^4) + \alpha_o H_a - \sigma \epsilon_o T_o^4 = 0.$$ 

The unknown potentials, and ultimately the heat flows are then linear functions of the data. In particular

$$q_l = c_1 (\sigma T_R^4) + c_2 (q_H) + c_3 (H_a),$$

where the $c_i$ depend on the various emissivity / absorptivity values.

On this region, the net heat flow to the RI material is given by:

$$S_2 = q_l (\sigma T_R^4, q_H, H_a) + C_b (T_1(t, z, \theta) - T_2(t, z, \theta)).$$

(19)

• $\Omega_3$: 
As in $\Omega_1$, here the RI material is conductively coupled to the outer right ring (above), and to the bladder (below) leading to

$$S_2(t, z, \theta) = C_{H3} (T_{o3}(t) - T_2(t, z, \theta)) + C_b (T_1(t, z, \theta) - T_2(t, z, \theta)).$$

(20)
On the $z$-boundaries we impose homogeneous Neumann boundary conditions, \textit{viz},
\begin{equation}
\nabla_{\hat{n}}T_i \cdot \hat{n} = \frac{\partial T_i(z, \theta)}{\partial z} = 0, \quad z = \pm L/2, \quad 0 \leq \theta \leq 2\pi,
\end{equation}
while on the $\theta$-boundaries we impose periodicity
\begin{equation}
T_i(t, z, 0) = T_i(t, z, 2\pi), \quad -L/2 \leq z \leq L/2.
\end{equation}

### 3.5.3 Bulk Temperature Models

The five \textit{bulk} temperatures ($\bar{T}$) evolve according to Equation (3). It’s necessary to identify the capacitance ($C$) and the source term ($Q$) for each of these entities.

**Thermal Capacitances:**

- **Gas ($T_g$):**
  The nitrogen gas density, estimated at one-third standard atmospheric pressure and at $70^\circ$C, is 0.00034 g/cc. Based on an inflated volume of 25,700 cc the mass is about 9 g, and the thermal capacitance is $C_g = 6.5 \ J/^\circ$C.

- **Inner left (in-board) end-cap ($T_{il}$):**
  The end-cap elements are fabricated from an aluminum alloy (Al 6061 - T651, density = 2.7 g/cc, specific heat = .896 J/g/$^\circ$C ). The inner-left element consists of an inboard cap ($V = 38.42330$ cc) and an inner ring ($V = 21.69744$ cc). Hence, $C_{il} = 145.4 \ J/^\circ$C.

- **Inner right (out-board) end-cap ($T_{ir}$):**
  The inner-right element consists of an out-board cap ($V = 54.58135$ cc) and an inner ring ($V = 21.69744$ cc). Hence, $C_{ir} = 184.5 \ J/^\circ$C.

- **Outer left (in-board) or right (out-board) end-cap ($T_{ol}$, $T_{or}$):**
  The left and right outer elements are identical and consist of an Outer Ring ($V = 14.89039$ cc) and an Test End Fitting ($V = 242.60168$ cc). Hence, $C_{ol} = C_{or} = 622.9 \ J/^\circ$C.

**Heat Sources:**

- **Gas ($T_g$):**
  Thermal energy flows into the gas at the bladder and (inner) end-cap
boundaries. The resulting model is (c.f. Equation (16)):

\[
Q_g(t) = \int_{-L'}^{L'} \int_{-\pi}^{\pi} C_g \left[ T_1(t, z, \theta) - T_g(t) \right] \, a \, d\theta \, dz \\
+ \pi a^2 C_g \left[ (T_{il}(t) - T_g(t)) + (T_{ir}(t) - T_g(t)) \right].
\]

• Inner left (in-board) end-cap (\(T_{il}\)):
The inner-left end-cap exchanges thermal energy with the left-most portion of the bladder (c.f. Equation (15)) and, through radiant exchange, with the outer left end-cap. The radiant exchange is modeled as gray-body radiation between identical, parallel surfaces of area \(A_e\), and emissivity \(\epsilon_e\).

\[
Q_{il}(t) = \int_{-L'}^{-L/2} \int_{-\pi}^{\pi} C_e \left( T_1(t, z, \theta) - T_{il}(t) \right) \, a \, d\theta \, dz + \frac{\sigma \epsilon_e A_e}{2 - \epsilon_e} \left[ T_{il}^4(t) - T_{il}^4(t) \right].
\]

• Inner right (out-board) end-cap (\(T_{ir}\)):
The inner-right end-cap exchanges thermal energy with the right-most portion of the bladder (c.f. Equation (15)) and, through radiant exchange, with the outer left end-cap.

\[
Q_{ir}(t) = \int_{L/2}^{L} \int_{-\pi}^{\pi} C_r \left( T_1(t, z, \theta) - T_{ir}(t) \right) \, a \, d\theta \, dz \\
+ \frac{\sigma \epsilon_e A_e}{2 - \epsilon_e} \left[ T_{ir}^4(t) - T_{ir}^4(t) \right].
\]

• Outer left (in-board) end-cap (\(T_{ol}\)):
The outer end-cap element exchanges radiantly with the inner end-cap, and by conduction with the RI material. \(A_\Delta\) is the area of the outer bond.

\[
Q_{ol}(t) = \frac{\sigma \epsilon_e A_e}{2 - \epsilon_e} \left[ T_{il}^4(t) - T_{ol}^4(t) \right] \\
+ \frac{C_{H1}}{C_R + C_{H1}} \int_{A_\Delta} \left[ S_H + C_R(T_2(t, z, \theta) - T_{ol}(t)) \right] \, a \, d\theta \, dz.
\]

• Outer right (out-board) end-cap (\(T_{or}\)):
The right element is similar to the left:

\[ Q_{or}(t) = \frac{\sigma \epsilon_e A_e}{2 - \epsilon_e} \left[ T_{ir}^4(t) - T_{or}^4(t) \right] + \frac{C_{H3}}{C_R + C_{H3}} \int_{A_\Delta} \left[ S_H + C_R(T_2(t, z, \theta) - T_{or}(t)) \right] \, a \, d\theta \, dz . \]

4 Weak Form

To develop the weak-form of the model we begin by multiplying (1) by a test function \( \psi \) and integrating over the domain \( \Omega \). The first term on the right-hand side yields

\[
\int_{\Omega} \left( k_a \frac{\partial^2 T}{\partial z^2} + k_c \frac{\partial^2 T}{\partial \theta^2} \right) \phi(z, \theta) \, d\Omega = - \int_{\Omega} \left( \frac{\partial \phi}{\partial z} k_a \frac{\partial T}{\partial z} + \frac{\partial \phi}{\partial \theta} k_c \frac{\partial T}{\partial \theta} \right) \, d\Omega + \int_{\Gamma} \left[ \sqrt{k_a \hat{e}_z} \frac{\partial T}{\partial z} + \sqrt{k_c \hat{e}_\theta} \frac{\partial T}{\partial \theta} \right] \cdot \hat{n} \, d\Gamma .
\]

The last term vanishes by virtue of the homogeneous Neumann boundary condition (21).

The contribution from the source terms on the right-side of (1) is given (formally) by:

\[
\int_{\Omega} \frac{S(t, z, \theta)}{\delta} \phi(z, \theta) \, d\Omega ,
\]

while each of the lumped components evolves as

\[
C_k \frac{dT_k(t)}{dt} = Q_k(t) ,
\]

where \( k \) assumes each of the five values \( \{ g, i\ell, o\ell, ir, or \} \), and expressions for each of the source terms (\( Q \)) have been listed above.

5 Numerical Approximations

5.1 General Ideas

Our numerical implementation is based on a finite-element (Galerkin) formulation to approximate the spatial distribution and a backward Euler for the time-derivative. Accordingly, we have

\[
T(t^n, z, \theta) \approx \sum_{j=1}^{N} T^n_j \psi_j(z, \theta) .
\]

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The basis functions \((\psi_j)\) are also used for the test functions \((\phi)\), so that the first term in the weak formulation yields

\[
\int_{\Omega} \left( \frac{\partial \psi_i}{\partial z} k_a \frac{\partial T(t_n)}{\partial z} + \frac{\partial \psi_i}{\partial \theta} k_c \frac{\partial T(t_n)}{\partial \theta} \right) d\Omega \approx \sum_{j=1}^{N} T_j^n \left[ \int_{\Omega} \left( \frac{\partial \psi_i}{\partial z} k_a \frac{\partial \psi_j}{\partial z} + \frac{\partial \psi_i}{\partial \theta} k_c \frac{\partial \psi_j}{\partial \theta} \right) d\Omega \right]
\]

\(i = 1, \ldots, N\).

In discrete form the unsteady model (2) leads to a system of nonlinear equations for the unknowns \((T_{j}^{n+1})\)

\[
\sum_{j=1}^{N} \rho C_p \frac{T_{j}^{n+1} - T_{j}^{n}}{\delta t} \int_{\Omega} \psi_j(z, \theta) \psi_i(z, \theta) d\Omega
+ \sum_{j=1}^{N} T_{j}^{n+1} \left[ \int_{\Omega} \left( \frac{\partial \psi_i}{\partial z} k_a \frac{\partial \psi_j}{\partial z} + \frac{\partial \psi_i}{\partial \theta} k_c \frac{\partial \psi_j}{\partial \theta} \right) d\Omega \right]
- \frac{1}{\delta} \int S(T_{j}^{n+1}, z, \theta) \psi_i(z, \theta) d\Omega = 0 \quad i = 1, 2, \ldots, N.
\] (24)

Equation (24) is augmented with backward Euler approximations for each of the five ordinary differential equations (23).

5.2 A Triangular Covering

The specific spatial discretization covers the region \(\Omega\) with triangles and employs linearly varying (spatial) functions on each triangle. The cylindrical surface is unwrapped to the rectangular \((z, y)\) region \([-L/2, L/2] \times [0, 2\pi R]\), where \(L\) is the length of the tube and \(R\) is its radius. As noted above, the sub-regions \((\Omega_1, \Omega_2, \Omega_3)\) are given by:

- \(\Omega_1 = [-L/2, -L/2 + \Delta] \times [0, 2\pi R]\),
- \(\Omega_2 = [-L/2 + \Delta, L/2 - \Delta] \times [0, 2\pi R]\),
- \(\Omega_3 = [L/2 - \Delta, L/2] \times [0, 2\pi R]\).

In our application the bond length \(\Delta\) is quite small compared to the overall length \(L\) (\(\Delta = 15\) mm, \(L = 1998\) mm). Accordingly, we construct a non-uniform \(z\)-grid, with \(z_1 = -L/2, z_2 = -L/2 + \Delta, z_{n_z-1} = L/2 - \Delta, z_{n_z} = L/2\). The intermediate points are located so that the grid spacing varies smoothly. The \(y\)-grid is uniform.
Based on this Cartesian grid we construct a triangular mesh over the region $\Omega$. Note, that by construction no triangular element overlaps a $\Omega_i, \Omega_j$ boundary. A typical result for $nz = 21, ny = 13$ is shown in Figure 4. The narrow strip of $24 (2 (ny-1))$ triangles on the left (resp., right) edge covers the region $\Omega_1$ (resp., $\Omega_3$). There are $nz \times ny$ nodal points, but periodic boundary conditions match the $(nz - 2)$ interior points along the $j = 1$, and $j = ny$ boundaries, so that there are $nz \times (ny - 1) + 2$ unknowns for each temperature field (RI and bladder). Including the five temperatures for the discrete components, we have $N = 2 \times nz \times (ny - 1) + 9$ unknowns. For the case illustrated in Figure 4 this evaluates to 513 unknowns.

5.3 Interior Radiation: Some Details

As noted in Subsection 3.1, radiant exchange interior to the cylinder is modeled by the radiosity distribution which must satisfy the integral equation (12). Here we use a rather crude approximation, wherein pairs of triangles are combined to generate a rectangular cover for $\Omega_2$. The radiosity field is approximated by a piecewise constant function on each rectangle (and on the left/right inner end-cap surfaces), while the integrals in equations (11, 12) are evaluated by a one-point rule. This is effectively the procedure described in [5, see pp 86-92].

The radiant energy leaving any (finite) surface in an enclosure must arrive at one of the surfaces defining the closed domain. Thus, various terms (the integral view factors) in equations (11, 12) must sum to unity. Additionally, the Second Law imposes a reciprocity between view factor pairs. Since inexact numerical integration can violate these conditions we use Lawson’s procedure[3] to correct the inaccuracies. With these approximations the (piecewise constant) radiosity function can be computed by a matrix
operation

\[ B = M(\sigma \hat{T}^4) , \]

where \( \hat{T} \) is a column vector of \((nz - 3) \times (ny - 1) + 2\) values of average temperatures for each rectangular elements (plus one for each end-cap). In fact we are actually interested in the net radiant flux on each surface given by (5). Simple substitution leads to the implementation

\[ S = \left[ \frac{\alpha M - \epsilon I}{1 - \alpha} \right] (\sigma \hat{T}^4) \overset{\triangle}{=} \hat{M}(\sigma \hat{T}^4) . \quad (25) \]

### 5.4 Solution Procedure

The nonlinear system of equations (24, plus ODE’s) is solved by a Newton iteration, globalized by a step-length-selection rule. In fact, the time-step and error tolerances are selected so that the Newton iteration (almost always) converges in a single step. Where convenient, efficiency is realized in constructing the Jacobian. For example, by virtue of the explicit form for the internal radiation (25), its contributions to the Jacobian can be readily computed. The numerical procedure have been implemented in a MATLAB code.

### References


